USE OF SPLINES IN THE SOLUTION OF INVERSE BOUNDARY PROBLEMS OF HEAT CONDUCTION

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A stable algorithm is proposed for the solution of one-dimensional inverse boundary problems of heat conduction based on the solution of the Cauchy problem. The incorrectness of the problem is eliminated by the use of regularized splines.

Inverse boundary problems of heat conduction, as is known [1], belong to the class of incorrectly stated problems. A rather full survey of the publications on the methods of their solution is contained in [2-3]. In the present report to construct a stable algorithm for the solution of inverse boundary problems we use the solution of the Cauchy problem, which for the case of a plate in a linear formulation has the form [4]

$$t(f;\ q;\ x,\ \tau)=\sum_{n=0}^{N}\frac{\left[\frac{(x-x^*)^2}{a}\right]^n}{(2n)!}f^{(n)}(\tau)-\frac{x-x^*}{\lambda}\sum_{n=0}^{N}\frac{\left[\frac{(x-x^*)^2}{a}\right]^n}{(2n+1)!}q^{(n)}(\tau)\quad (N=\infty), \tag{1}$$

where f and q are functions of the temperature and heat flux, respectively, at the point x^*. The latter are assumed to be analytical in their regions of definition. It is easy to show that in this case the solution (1) converges uniformly in a finite region. Consequently, instability of (1) against small perturbations of the functions f and q is caused only by the differentiation operator. As is known [2], at points of a body remote from the heat-transfer surface the temperature and heat flux are represented by smooth functions. We therefore limit the sums in the solution (1) to three terms (N = 2). Then to obtain a stable approximate solution to an inverse problem it is sufficient to determine stable algorithms for numerical differentiation of up to second order inclusively. We will construct the latter on the basis of the regularization method [1] using spline functions. Such an approach was discussed in [5]. In the present report we propose a different regularization algorithm, making it possible to solve inverse problems of heat conduction in various formulations.

Suppose that the function $g \in C^2[a,\ b]$ on the grid

$$\omega=\{a=\tau_0<\tau_1<\ldots<\tau_k=b\}$$

satisfies the equation

$$Lg=u, \tag{2}$$

where u is the grid function on ω, known with an error $\Delta=\{\delta_0,\ \delta_1,\ldots,\ \delta_k\}$; L is a differential operator of no higher than second order.

Let F be a set of functions $g \in C^2[a,\ b]$ satisfying the limits

$$|Lg-u|\leqslant\Delta \tag{3}$$

on the grid ω. Then, in accordance with [1], the regularized solution of the problem of differentiation of the function g satisfying Eq. (2) will be the element $\bar{g}$ which minimizes the functional

$$\Omega[g]=\int_a^b\left(\frac{d^2g}{d\tau^2}\right)^2 d\tau\to\min \tag{4}$$

in the set F. It is shown in [6] that a cubic spline yields the minimum to the functional Ω in F. Consequently, the functions

$$g(\tau)\equiv S_\omega(v;\ \tau), \tag{5}$$

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where the right side is a cubic spline interpolating the grid function v on ω, will be subjected to analysis.

As is known, a cubic spline is determined by a system of linear equations with a three-diagonal matrix, for the solution of which a very efficient algorithm was presented in [6]. The latter allows one to successfully realize the variational problem (3)-(5) by the method of local variations with respect to v [7]. The calculation of the values of the functional $\Omega[g]$ with allowance for (5) which is required in the process is reduced to the elementary summation of squares.

Any grid function v^0 for which $S_\omega(v^0; \tau) \in F$ is taken as the zeroth approximation of the solution of the problem (3)-(5). If such is not known it can be found by solving the problem

$$\int_a^b |L g - u|^2 d\tau \to \min_v \qquad (6)$$

up to the fulfillment of the conditions (3). The realization of the problem (6) is accomplished by the method of local variations without limitations, starting with an arbitrary function v.

The boundary conditions for the splines, if they are not known, can be assigned in the form

$$g'(a) = 0; \quad g'''(b) = 0,$$

which corresponds to a constant initial temperature distribution and an established thermal regime, respectively.

It should be noted that the problem (3)-(5) uniquely defines a function $g''(\tau)$, and the function $g'(\tau)$ is reconstructed with the accuracy of a constant belonging to the set F. The convergence of the third derivative of a cubic spline [6] allows one to estimate the discarded terms of the series (1) from the equations

$$R_{t_w} = \frac{\left(\frac{l^2}{a}\right)^3}{720}\left[f'''(\theta_1) - \frac{l}{7\lambda} q'''(\theta_2)\right],$$

$$R_{q_w} = -\frac{\lambda}{l}\,\frac{\left(\frac{l^2}{a}\right)^3}{120}\left[f'''(\theta_1) - \frac{l}{6\lambda} q'''(\theta_2)\right],$$

where l is the distance from the point x^* to the boundary of the body at which the temperature t_w and heat flux q_w are reconstructed; θ_1 and θ_2 are some points out of the time interval under consideration.

Let us apply the described algorithm for numerical differentiation to the solution of concrete inverse boundary problems with constant thermophysical parameters.

§1. The uniqueness of the inverse problem for a plate $x_1 \le x \le x_2$ is specified by the values of the temperature and heat flux at the boundary x_1:

$$t(x_1, \tau_j) = t_j, \quad j = 0, 1, \ldots, k; \qquad (7)$$

$$-\lambda \frac{\partial t(x_1, \tau_j)}{\partial x} = q_j, \quad j = 0, 1, \ldots, k, \qquad (8)$$

with the equalities (7) and (8) being known with errors Δ_1 and Δ_2, respectively.

To solve the problem we use the solution of the Cauchy problem (1) with $x^* = x_1$. Substituting (1) into (7), we obtain $f(\tau_j) = t_j$, $j = 0, 1, \ldots, k$, on ω.

Thus, in the given case the operator L is of zeroth order. From (3), with allowance for (5), we obtain the conditions of affiliation of the function g to the set F:

$$|v_j - t_j| \leqslant \delta_{1j}, \quad j = 0, 1, \ldots, k. \qquad (9)$$

Consequently, the functions f' and f'' are reconstructed through the solution of the variational problem (4)-(5)-(9). The testing of the conditions (9) is elementary and, in addition, it is done only at that grid node at which the function v is varied. As the zeroth approximation we take $v_j^0 = t_j$, $j = 0, 1, \ldots, k$.

Proceeding in the same way for the equality (8), we arrive at an analogous variational problem for the function $q(\tau)$. In this case the condition of affiliation to the set F will be

$$|v_j - q_j| \leqslant \delta_{2j}, \quad j = 0, 1, \ldots, k. \qquad (10)$$

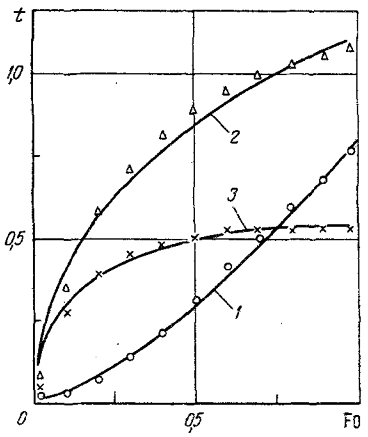

Fig. 1

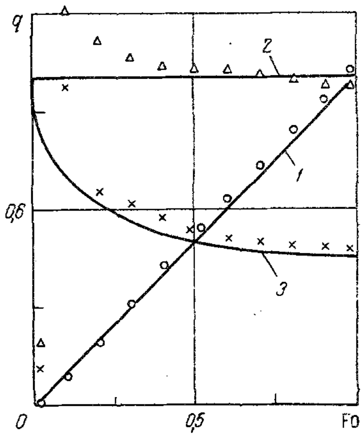

Fig. 2

Fig. 1. Variation of temperature of heat-supply surface for different variants of model problems: solid curves) exact values of t_W; points) values of t_W reconstructed through solution of model inverse problems (δ = 0.05).

Fig. 2. Variation of heat flux through heat-supply surface for different variants of model problems: solid curves) exact values of q_W; points) values of q_W obtained through solution of model inverse problems (δ = 0.05).

Thus, the unknown values of the temperature and heat flux at the boundary x_2 are

$$t_w = t(\bar{f};\ \bar{q};\ x_2,\ \tau),\ q_w = -\lambda \frac{\partial}{\partial x}\ t(\bar{f};\ \bar{q};\ \ x_2,\ \tau). \tag{11}$$

Here $\bar{f}$ is the solution of the variational problem (4)-(5)-(9); $\bar{q}$ is the solution of the problem (4)-(5)-(10).

§2. The uniqueness of the inverse boundary problem for a plate $x_1 \leq x \leq x_2$ is specified by the values of the temperatures at internal points of the body:

$$t(x_3,\ \tau_j) = t_{1j},\quad j = 0,\ \ldots,\ k;\quad x_1 \leqslant x_3 < x_2, \tag{12}$$

$$t(x_4,\ \tau_j) = t_{2j},\quad j = 0,\ \ldots,\ k;\quad x_3 < x_4 < x_2, \tag{13}$$

with errors Δ_1 and Δ_2, respectively. Substituting (1) with $x^* = x_3$ into (12) and arguing as in Sec. 1, we arrive at a variational problem analogous to (4)-(5)-(9). Let $\bar{f}$ be the solution of the latter; then $t(\bar{f};\ q;\ x,\ \tau)$ together with (13) determines a second-order differential equation relative to the function q on the grid ω. In this case the condition of affiliation of the function g to the set F has the form

$$|t(\bar{f};\ g;\ x_4,\ \tau_j) - t_{2j}| \leqslant \delta_{2j},\quad j = 0,\ \ldots,\ k. \tag{14}$$

The conditions (14) must be tested at all nodes of the grid ω with local variation of the function v. The initial approximation v^0 of the solution to the problem (4)-(5)-(14) is obtained by solving the problem (6). If $\bar{q}$ is the solution of the problem (4)-(5)-(14), then the solution of the inverse problem is given by Eqs. (11).

Inverse problems in other formulations of the conditions of uniqueness can be solved in an analogous way.

In the examples cited the boundary of the body at which the temperatures t_W and heat fluexes q_W are reconstructed can move in accordance with a given law.

We note that the proposed algorithm for the approximate solution of inverse boundary problems is also applicable for bodies of more complex shape and for variable thermophysical parameters. For this it is only necessary to determine the approximate solution to the corresponding Cauchy problem. And the algorithm for the numerical differentiation remains as before. The method for constructing an approximate solution to the Cauchy problem for a plate with variable thermophysical properties is presented in [4].

The described algorithm for the solution of inverse problems was realized in the ALGOL language for a BESM-4M digital computer. We made systematic calculations on the reconstruction of the conditions at the

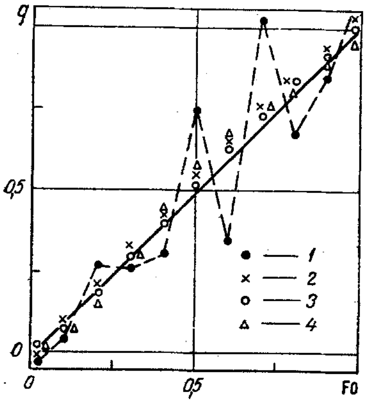

Fig. 3. Results of reconstruction of heat flux with different perturbations of the initial data (variant 1): solid line) exact values of q_w; 1) q_w reconstructed without regularization of the derivative (δ = 0.05); 2) q_w reconstructed by the proposed algorithm (δ = 0.05); 3) the same for δ = 0.01; 4) for δ = 0.1.

heat-supply surface of an infinite plate $x_1 \le x \le x_2$ with constant thermophysical properties. Heat removal from the surface x_1 took place by emission. The heating was carried out through heat supply to the surface x_2 with a given intensity q_w. The problem was solved in dimensionless quantities by the finite-difference method for a constant initial temperature distribution.

The inverse problem was studied in the formulation (7)-(8). The right sides of the equations, which determine the uniqueness of the solution of the problem, were perturbed in accordance with the equation

$$t_j = \bar{t}_j(1 + \delta\varepsilon_j), \quad j = 1, \ldots, N-1.$$

Here $\bar{t}_j$ is the exact value of the function; ε_j is a random quantity uniformly distributed over the segment [−1, 1].

In Figs. 1 and 2 we present the results of calculations in one of the realizations of the perturbations of the conditions (7)-(8) (δ = 0.05) for three variants: 1) q_w = Fo; 2) q_w = 1; 3) $q_w = 1 - t_w$. As is seen, there is satisfactory reconstruction of the temperatures and heat fluxes at the boundary everywhere except for a small initial section for variants 2 and 3. The latter is evidently a consequence of the discontinuity of the heat-flux function at the boundary at the initial time.

Figure 3, in which the reconstructed values of the heat flux for the first variant with different δ, as well as the values of q_w obtained without regularization of the derivatives, are presented clearly demonstrates the stability of the proposed algorithm for the solution of inverse boundary problems of heat conduction.

The time of calculation of one realization of the inverse problem with k = 20 did not exceed 6 min.

NOTATION

t	is the temperature;
x	is the current coordinate;
τ	is the current time;
a	is the coefficient of thermal diffusivity;
λ	is the coefficient of thermal conductivity;
$k + 1$	is the number of nodes of grid ω.

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A SCHEME OF FRACTIONAL STEPS FOR A NONSTEADY INTERNAL CONJUGATE PROBLEM OF HEAT TRANSFER IN FLOW OF AN INCOMPRESSIBLE LIQUID WITH VARIABLE THERMOPHYSICAL PROPERTIES

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The conjugate problem of heat transfer during the non steady laminar flow of a viscous incompressible liquid at the entrance section of a plane, annular, or cylindrical channel or in a closed region is discussed.

For the calculation of transitional processes in the flow of cryogenic and high-temperature liquids in channels, for the calculation of transitional processes under conditions of free and free–forced convection in channels and closed regions, etc. it is necessary to create methods for the solution of internal conjugate problems of heat exchange allowing for the nonsteadiness and two-dimensionality of the processes of flow and heat transfer and the true temperature dependence of the properties of the liquid and the wall materials. The application of analytical methods for the solutions of conjugate problems in such a formulation is difficult. An economical, convergent, nonlinear, difference scheme which approximates the stated problem is suggested in the present report.

The nonsteady two-dimensional laminar flow of a viscous incompressible liquid in a plane, annular, or cylindrical channel is analyzed. The viscosity, heat capacity, and thermal conductivity of the liquid depend in a known way on the temperature, the density of the walls depends on the coordinates, and the heat capacity and thermal conductivity depend on the coordinates and the temperature. Heat release occurs in the channel walls and in the liquid. The amount of heat released per unit time per unit mass is a known function of the coordinates and time. A mass force, which depends on the coordinates, time, and the temperature acts on the liquid. The temperature distribution over the ends and outer surfaces of the channel walls is known and varies with time. The pressure in the channel varies continuously and at the exit it equals the pressure of the surrounding medium, which depends on time in a known way. At the contact surfaces between the liquid and the walls a coolant is supplied, the rate of inflow of which is known, while the enthalpy depends on the temperature in a known way. It is assumed that before the start of the process a known steady flow of liquid existed in the channel with a known temperature distribution for the liquid and the walls. At the starting time some perturbation of the velocity and temperature is supplied to the entrance, and heat release and the inflow of coolant begin. The nonsteady process which develops is analyzed. The conditions of temperature conjugation are set up at the liquid–wall contact surfaces in the form of boundary conditions of the fourth kind. To set up the boundary conditions at the exit cut of the channel, simplifying assumptions are made. It is assumed that the channel is long enough, and the coolant supply and the heat sources are concentrated in the entrance section, so that the flow becomes one-dimensional near the exit. It is also assumed that the longitudinal heat flux, due to the heat conduction of the liquid, can be estimated and assigned in the form of a known function of time near the exit.

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